Section 7.4 – Trigonometric Identities

Recall the definition of an identity:

Two functions f and g are said to be **identically equal** if $f(x) = g(x)$ for every value of x for which both functions are defined. Such an equation is referred to as an **identity**. An equation that is not an identity is called a **conditional equation**.

Example 1:

Identities: (True for all values of x in the domain)

$$\sin^2 x + \cos^2 x = 1$$

$$\sec x = \frac{1}{\cos x}$$

$$(x-2)^2 = x^2 - 4x + 4$$

Conditional Equations:

$\sin x = 0$ This is true only if $x = k\pi$, for k an integer.

$x + 3 = 0$ This is true only if $x = -3$.

The following summarizes the trigonometric identities that have already been established:

Quotient Identities

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \cot\theta = \frac{\cos\theta}{\sin\theta}$$

Reciprocal Identities

$$\csc\theta = \frac{1}{\sin\theta} \qquad \sec\theta = \frac{1}{\cos\theta} \qquad \cot\theta = \frac{1}{\tan\theta}$$

Pythagorean Identities

$$\sin^2\theta + \cos^2\theta = 1 \qquad \tan^2\theta + 1 = \sec^2\theta \qquad 1 + \cot^2\theta = \csc^2\theta$$

Even-Odd Identities

$$\sin(-\theta) = -\sin\theta \qquad \cos(-\theta) = \cos\theta \qquad \tan(-\theta) = -\tan\theta$$

$$\csc(-\theta) = -\csc\theta \qquad \sec(-\theta) = \sec\theta \qquad \cot(-\theta) = -\cot\theta$$

This list comprises what shall be referred to as the **basic trigonometric identities**. You need to know these relationships and be comfortable with variations of them.

Use Algebra to Simplify Trigonometric Expressions

The ability to use algebra to manipulate trigonometric expressions is a key skill that one must have to establish identitites. Four basic algebraic techniques are used to establish identities:

1) Rewriting a trigonometric expression in terms of sine and cosine only
2) Multiplying the numerator and denominator of a ratio by a “well-chosen 1”
3) Writing sums of trigonometric ratios as a single ratio
4) Factoring

Example 2: a) Simplify $\frac{\csc\theta}{\cot\theta}$ by rewriting each trigonometric function in terms of sine and cosine functions.

$$\frac{\csc\theta}{\cot\theta} = \frac{\frac{1}{\sin\theta}}{\frac{\cos\theta}{\sin\theta}}$$

$$= \frac{1}{\sin\theta}\left(\frac{\sin\theta}{\cos\theta}\right)$$

$$= \frac{1}{\cos\theta}$$

$$= \sec\theta$$

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Example 2: b) Show that $\frac{\cos\theta}{1-\sin\theta}=\frac{1+\sin\theta}{\cos\theta}$ by multiplying the numerator and denominator by $1+\sin\theta$.

$$\frac{\cos\theta}{1-\sin\theta}=\frac{\cos\theta}{1-\sin\theta}\left(\frac{1+\sin\theta}{1+\sin\theta}\right)$$
$$=\frac{\cos\theta(1+\sin\theta)}{1-\sin^2\theta}$$
$$=\frac{\cos\theta(1+\sin\theta)}{\cos^2\theta}$$
$$=\frac{1+\sin\theta}{\cos\theta}$$

c) Simplify $\frac{\cos\theta}{\sin\theta}+\frac{\sin\theta}{\cos\theta}$ by rewriting the expression as a single ratio

$$\frac{\cos\theta}{\sin\theta}+\frac{\sin\theta}{\cos\theta}=\left(\frac{\cos\theta}{\sin\theta}\right)\frac{\cos\theta}{\cos\theta}+\left(\frac{\sin\theta}{\cos\theta}\right)\frac{\sin\theta}{\sin\theta}$$
$$=\frac{\cos^2\theta}{\sin\theta\cos\theta}+\frac{\sin^2\theta}{\sin\theta\cos\theta}$$
$$=\frac{\sin^2\theta+\cos^2\theta}{\sin\theta\cos\theta}$$
$$=\frac{1}{\sin\theta\cos\theta}$$
$$=\frac{\sec\theta}{\sin\theta}$$

d) Simplify $\frac{\cos^2\theta-1}{\tan\theta\cos\theta+\tan\theta}$ by factoring

$$\frac{\cos^2\theta-1}{\tan\theta\cos\theta+\tan\theta}=\frac{(\cos\theta+1)(\cos\theta-1)}{\tan\theta(\cos\theta+1)}$$
$$=\frac{\cos\theta-1}{\tan\theta}$$

Establish Identities

Many of the problems in this chapter ask you to "Establish the Identity … . " To establish an identity, you need to start with one side of the given identity and use basic trigonometric identities and algebraic manipulations to obtain the other side. Keep your goal in mind! As you manipulate one side of the expression, keep in mind the form of the expression you want on the other side. Be careful not to handle identities to be established as if they were equations. You cannot establish an identity by such methods as adding the same expression to each side and obtaining a true statement. This practice is not allowed, because the original statement is precisely the one that you are trying to establish. You do not know until it has been established that it is, in fact, true. There is often more than one way to establish an identity. The selection of appropriate basic identities to obtain the desired result is learned only through experience and lots of practice.

A graphing utility cannot be used to establish an identity. Identities must be established algebraically. Graphing just provides evidence of an identity.

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Example 3: Establish the identity $\csc\theta\cos\theta = \cot\theta$

$$\csc\theta\cos\theta = \left(\frac{1}{\sin\theta}\right)\cos\theta \quad \text{using a reciprocal identity}$$

$$= \frac{\cos\theta}{\sin\theta}$$

$$= \cot\theta \quad \text{by the quotient identities}$$

$$\therefore \csc\theta\cos\theta = \cot\theta$$

Example 4: Establish the identity $1 + \tan^2(-\theta) = \sec^2\theta$

$$1 + \tan^2(-\theta) = 1 + (-\tan\theta)^2 \quad \text{since tan is odd}$$

$$= 1 + \tan^2\theta$$

$$= \sec^2\theta \quad \text{by the Pythagorean identities}$$

$$\therefore 1 + \tan^2(-\theta) = \sec^2\theta$$

Guidelines for establishing identities:

1) It is almost always preferable to start with the side containing the more complicated expression.
2) Rewrite sums or differences of quotients as a single quotient, especially if the other side contains only one term.
3) Sometimes rewriting one side in terms of sines and cosines only will help.
4) Always keep your goal in mind. As you manipulate one side of the expression, you must keep in mind the form of the expression on the other side.

Example 5: Establish the identity $\tan\theta\cot\theta - \cos^2\theta = \sin^2\theta$

$$\tan\theta\cot\theta - \cos^2\theta = \frac{\sin\theta}{\cos\theta}\left(\frac{\cos\theta}{\sin\theta}\right) - \cos^2\theta \quad \text{by the quotient identities}$$

$$= 1 - \cos^2\theta$$

$$= \sin^2\theta \quad \text{by the Pythagorean identities}$$

$$\therefore \tan\theta\cot\theta - \cos^2\theta = \sin^2\theta$$

Example 6: Establish the identity $\sec\theta - \tan\theta = \dfrac{\cos\theta}{1 + \sin\theta}$

$$\sec\theta - \tan\theta = \frac{1}{\cos\theta} - \frac{\sin\theta}{\cos\theta} \quad \text{by the quotient and reciprocal identities}$$

$$= \frac{1 - \sin\theta}{\cos\theta}$$

$$= \left(\frac{1 - \sin\theta}{\cos\theta}\right)\left(\frac{1 + \sin\theta}{1 + \sin\theta}\right) \quad \text{multiply by a factor of one}$$

$$= \frac{1 - \sin^2\theta}{\cos\theta\,(1 + \sin\theta)}$$

$$= \frac{\cos^2\theta}{\cos\theta(1 + \sin\theta)} \quad \text{by the Pythagorean identities}$$

$$= \frac{\cos\theta}{1 + \sin\theta}$$

$$\therefore \sec\theta - \tan\theta = \frac{\cos\theta}{1 + \sin\theta}$$

For more examples, look at the book's examples on pages 473-475.

All material has been taken from *Precalculus,* by M. Sullivan, 10th Edition