Section 7.5 – Sum and Difference Formulas – Day 1

This section continues the derivation of trigonometric identities by obtaining formulas that involve the sum or difference of two angles, such as $\cos(\alpha + \beta)$ and $\sin(\alpha - \beta)$. These formulas are called the sum and difference formulas.

Use Sum and Difference Formulas to Find Exact Values

One use of the Sum and Difference formulas is to obtain the exact value of sine and cosine functions of an angle that can be expressed as the sum or difference of angles whose sine and cosine are known exactly.

Theorem: Sum and Difference Formulas for Cosines

$$\cos(\alpha + \beta) = \cos\alpha \cos\beta - \sin\alpha \sin\beta$$
$$\cos(\alpha - \beta) = \cos\alpha \cos\beta + \sin\alpha \sin\beta$$

Example 1: Find the exact value of $\cos\left(\frac{7\pi}{12}\right)$.

$$\cos\left(\frac{7\pi}{12}\right) = \cos\left(\frac{4\pi}{12} + \frac{3\pi}{12}\right)$$
$$= \cos\left(\frac{\pi}{3} + \frac{\pi}{4}\right)$$
$$= \cos\left(\frac{\pi}{3}\right)\cos\left(\frac{\pi}{4}\right) - \sin\left(\frac{\pi}{3}\right)\sin\left(\frac{\pi}{4}\right) \quad \text{by the theorem above}$$
$$= \left(\frac{x}{r}\right)\left(\frac{x}{r}\right) - \left(\frac{y}{r}\right)\left(\frac{y}{r}\right) \quad \text{Now } \frac{\pi}{3} : (x, y) = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right) \text{ and } \frac{\pi}{4} : (x, y) = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right).$$
$$= \left(\frac{\frac{1}{2}}{1}\right)\left(\frac{\frac{\sqrt{2}}{2}}{1}\right) - \left(\frac{\frac{\sqrt{3}}{2}}{1}\right)\left(\frac{\frac{\sqrt{2}}{2}}{1}\right)$$
$$= \left(\frac{1}{2}\right)\left(\frac{\sqrt{2}}{2}\right) - \left(\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{2}}{2}\right)$$
$$= \frac{\sqrt{2}}{4} - \frac{\sqrt{6}}{4}$$
$$= \frac{1}{4}\left(\sqrt{2} - \sqrt{6}\right)$$

Example 2: Find the exact value of $\cos 80^\circ \cos 20^\circ + \sin 80^\circ \sin 20^\circ$.

$$\cos 80^\circ \cos 20^\circ + \sin 80^\circ \sin 20^\circ = \cos(80^\circ - 20^\circ) \quad \text{by the Sum and Difference Formula Theorem for Cosines}$$
$$= \cos 60^\circ$$
$$= \frac{x}{r} \quad \text{Now } 60^\circ : (x, y) = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right).$$
$$= \frac{\frac{1}{2}}{1}$$
$$= \frac{1}{2}$$

Theorem: Sum and Difference Formulas for Sines

$$\sin(\alpha + \beta) = \sin\alpha \cos\beta + \cos\alpha \sin\beta$$
$$\sin(\alpha - \beta) = \sin\alpha \cos\beta - \cos\alpha \sin\beta$$

Section 7.5 – Sum and Difference Formulas – Day 1 (continued)

Example 3: Find the exact value of $\sin\left(\frac{5\pi}{12}\right)$.

$$\sin\left(\frac{5\pi}{12}\right) = \sin\left(\frac{3\pi}{12} + \frac{2\pi}{12}\right)$$

$$= \sin\left(\frac{\pi}{4} + \frac{\pi}{6}\right)$$

$$= \sin\left(\frac{\pi}{4}\right)\cos\left(\frac{\pi}{6}\right) + \cos\left(\frac{\pi}{4}\right)\sin\left(\frac{\pi}{6}\right)$$ by the Sum Formula for Sines Theorem

$$= \left(\frac{y}{r}\right)\left(\frac{x}{r}\right) + \left(\frac{x}{r}\right)\left(\frac{y}{r}\right)$$ Now $\frac{\pi}{4} : (x,\ y) = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$ and $\frac{\pi}{6} : (x,\ y) = \left(\frac{\sqrt{3}}{2},\ \frac{1}{2}\right)$.

$$= \left(\frac{\frac{\sqrt{2}}{2}}{1}\right)\left(\frac{\frac{\sqrt{3}}{2}}{1}\right) + \left(\frac{\frac{\sqrt{2}}{2}}{1}\right)\left(\frac{\frac{1}{2}}{1}\right)$$

$$= \left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{3}}{2}\right) + \left(\frac{\sqrt{2}}{2}\right)\left(\frac{1}{2}\right)$$

$$= \frac{\sqrt{6}}{4} + \frac{\sqrt{2}}{4}$$

$$= \frac{1}{4}\left(\sqrt{6} + \sqrt{2}\right)$$

Example 4: Given $\sin\alpha = \frac{3}{5},\ 0 < \alpha < \frac{\pi}{2};\quad \cos\beta = \frac{2\sqrt{5}}{5},\quad \frac{-\pi}{2} < \beta < 0$. Determine $\sin(\alpha + \beta)$, $\sin(\alpha - \beta)$, and $\cos(\alpha + \beta)$.

Preliminary Work:

α is in quadrant I $\Rightarrow$ $\cos\alpha$ is positive

$$\cos\alpha = \sqrt{1 - \sin^2\alpha}$$

$$= \sqrt{1 - \left(\frac{3}{5}\right)^2}$$

$$= \sqrt{1 - \left(\frac{9}{25}\right)}$$

$$= \sqrt{\frac{16}{25}}$$

$$= \frac{4}{5}$$

β is in quadrant IV $\Rightarrow$ $\sin\beta$ is negative

$$\sin\beta = -\sqrt{1 - \cos^2\beta}$$

$$= -\sqrt{1 - \left(\frac{2\sqrt{5}}{5}\right)^2}$$

$$= -\sqrt{1 - \left(\frac{4(5)}{25}\right)}$$

$$= -\sqrt{1 - \left(\frac{4}{5}\right)}$$

$$= -\sqrt{\frac{1}{5}}$$

$$= \frac{-1}{\sqrt{5}}$$

$$= \frac{-1}{\sqrt{5}}\left(\frac{\sqrt{5}}{\sqrt{5}}\right)$$

$$= \frac{-\sqrt{5}}{5}$$

a) $\sin(\alpha + \beta) = \sin\alpha\ \cos\beta + \cos\alpha\ \sin\beta$

$$= \left(\frac{3}{5}\right)\left(\frac{2\sqrt{5}}{5}\right) + \left(\frac{4}{5}\right)\left(\frac{-\sqrt{5}}{5}\right)$$

$$= \frac{6\sqrt{5}}{25} - \frac{4\sqrt{5}}{25}$$

$$= \frac{2\sqrt{5}}{25}$$

b) $\sin(\alpha - \beta) = \sin\alpha\ \cos\beta - \cos\alpha\ \sin\beta$

$$= \left(\frac{3}{5}\right)\left(\frac{2\sqrt{5}}{5}\right) - \left(\frac{4}{5}\right)\left(\frac{-\sqrt{5}}{5}\right)$$

$$= \frac{6\sqrt{5}}{25} + \frac{4\sqrt{5}}{25}$$

$$= \frac{10\sqrt{5}}{25}$$

$$= \frac{2\sqrt{5}}{5}$$

Section 7.5 – Sum and Difference Formulas – Day 1 (continued)

c) $\cos(\alpha + \beta) = \cos\alpha \cos\beta - \sin\alpha \sin\beta$

$$= \left(\frac{4}{5}\right)\left(\frac{2\sqrt{5}}{5}\right) - \left(\frac{3}{5}\right)\left(\frac{-\sqrt{5}}{5}\right)$$

$$= \frac{8\sqrt{5}}{25} + \frac{3\sqrt{5}}{25}$$

$$= \frac{11\sqrt{5}}{25}$$

Theorem: Sum and Difference Formulas for Tangents

$$\tan(\alpha + \beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta} \qquad \tan(\alpha - \beta) = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}$$

These formulas can be used for all angles except for odd integer multiples of $\frac{\pi}{2}$.

Example 5: Find the exact value of $\tan 15^\circ$.

$$\tan 15^\circ = \tan(60^\circ - 45^\circ)$$

$$= \frac{\tan 60^\circ - \tan 45^\circ}{1 + \tan 60^\circ \tan 45^\circ}$$

Now $60^\circ : (x_1, y_1) = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ and

$45^\circ : (x_2, y_2) = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$

$$= \frac{\frac{y_1}{x_1} - \frac{y_2}{x_2}}{1 + \left(\frac{y_1}{x_1}\right)\left(\frac{y_2}{x_2}\right)}$$

$$= \frac{\left(\frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}}\right) - \left(\frac{\frac{\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}}\right)}{1 + \left(\frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}}\right)\left(\frac{\frac{\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}}\right)}$$

$$= \frac{\sqrt{3} - 1}{1 + \sqrt{3}(1)}$$

$$= \frac{\sqrt{3} - 1}{1 + \sqrt{3}}$$

$$= \left(\frac{\sqrt{3} - 1}{1 + \sqrt{3}}\right)\left(\frac{1 - \sqrt{3}}{1 - \sqrt{3}}\right)$$

$$= \frac{\sqrt{3} - 3 - 1 + \sqrt{3}}{1 - 3}$$

$$= \frac{2\sqrt{3} - 4}{-2}$$

$$= \frac{-2(2 - \sqrt{3})}{-2}$$

$$= 2 - \sqrt{3}$$

All material has been taken from *Precalculus,* by M. Sullivan, 10th Edition