Section 7.5 – Trigonometric Equations – Day 3

Use Sum and Difference Formulas Involving Inverse Trigonometric Functions

Example 10: Find the exact value of : $\cos\left(\sin^{-1}\left(\frac{1}{2}\right)+\cos^{-1}\left(\frac{3}{5}\right)\right)$

You want the cosine of the sum of two angles, $\alpha=\sin^{-1}\left(\frac{1}{2}\right)$ and $\beta=\cos^{-1}\left(\frac{3}{5}\right)$.

Then $\sin\alpha=\frac{1}{2}, \frac{-\pi}{2}\le\alpha\le\frac{\pi}{2}$ and $\cos\beta=\frac{3}{5}, 0\le\beta\le\pi$.

Since $\sin\alpha$ is positive, α is in Quad I. Since $\cos\beta$ is positive, β is in Quad I.

Use Pythagorean Identities to determine $\cos\alpha$ and $\sin\beta$.

$$\cos\alpha=\sqrt{1-\sin^2\alpha} \quad \text{and} \quad \sin\beta=\sqrt{1-\cos^2\beta} \quad \text{positive since each angle is in Quad I}$$

$$\cos\alpha=\sqrt{1-\left(\frac{1}{2}\right)^2}=\sqrt{1-\frac{1}{4}}=\sqrt{\frac{3}{4}}=\frac{\sqrt{3}}{2}$$

$$\sin\beta=\sqrt{1-\left(\frac{3}{5}\right)^2}=\sqrt{1-\frac{9}{25}}=\sqrt{\frac{16}{25}}=\frac{4}{5}$$

So,
$$\begin{aligned}\cos\left(\sin^{-1}\left(\frac{1}{2}\right)+\cos^{-1}\left(\frac{3}{5}\right)\right)&=\cos(\alpha+\beta)\\&=\cos\alpha\cos\beta-\sin\alpha\sin\beta\\&=\frac{\sqrt{3}}{2}\left(\frac{3}{5}\right)-\frac{1}{2}\left(\frac{4}{5}\right)\\&=\frac{3\sqrt{3}}{10}-\frac{4}{10}\\&=\frac{3\sqrt{3}-4}{10}\end{aligned}$$

Section 7.5 – Trigonometric Equations – Day 3 (continued)

Solve Trigonometric Equations Linear in Sine and Cosine

Sometimes it is necessary to square both sides of an equation to obtain expressions that allow the use of identities. Remember that squaring both sides of an equation may introduce extraneous solutions. So, you need to check apparent solutions.

Example 11A: Solve the equation: $\sin\theta + \cos\theta = 1, \quad 0 \le \theta < 2\pi$.

Method 1:

$$\sin\theta + \cos\theta = 1$$

$$\frac{1}{\sqrt{2}}\sin\theta + \frac{1}{\sqrt{2}}\cos\theta = \frac{1}{\sqrt{2}} \quad \text{Multiply by } \frac{1}{\sqrt{2}}$$

$$\frac{\sqrt{2}}{2}\sin\theta + \frac{\sqrt{2}}{2}\cos\theta = \frac{\sqrt{2}}{2}$$

If you let $\beta = \frac{\pi}{4}$, where $\frac{\pi}{4} : (x, y) = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$, then

$$\sin\beta = \sin\frac{\pi}{4} = \frac{y}{r} = \frac{\frac{\sqrt{2}}{2}}{1} = \frac{\sqrt{2}}{2} \quad \text{and} \quad \cos\beta = \cos\frac{\pi}{4} = \frac{x}{r} = \frac{\frac{\sqrt{2}}{2}}{1} = \frac{\sqrt{2}}{2}$$

So, $\frac{\sqrt{2}}{2}\sin\theta + \frac{\sqrt{2}}{2}\cos\theta = \frac{\sqrt{2}}{2}$

becomes $\cos\beta\,\sin\theta + \sin\beta\,\cos\theta = \frac{\sqrt{2}}{2}$.

$$\sin\theta\,\cos\beta + \cos\theta\,\sin\beta = \frac{\sqrt{2}}{2}$$

$$\Rightarrow \sin(\theta + \beta) = \frac{\sqrt{2}}{2}$$

$$\sin\left(\theta + \frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$$

Sine is positive in quadrants I and II, so the general solutions are

$\theta + \frac{\pi}{4} = \frac{\pi}{4} + 2k\pi$, k any integer and $\theta + \frac{\pi}{4} = \frac{3\pi}{4} + 2k\pi$, k any integer

$\Rightarrow \theta = 0 + 2k\pi$, k any integer and $\theta = \frac{2\pi}{4} + 2k\pi$, k any integer

$= \frac{\pi}{2} + 2k\pi$, k any integer

On the interval $[0, 2\pi)$, the solutions are $\theta = 0$ and $\theta = \frac{\pi}{2}$.

Section 7.5 – Trigonometric Equations – Day 3 (continued)

Example 11B: Solve the equation: $\sin\theta + \cos\theta = 1, \quad 0 \le \theta < 2\pi$.

Method 2:

$\sin\theta + \cos\theta = 1$

$(\sin\theta + \cos\theta)^2 = 1^2$

$\sin^2\theta + 2\sin\theta\cos\theta + \cos^2\theta = 1$

$2\sin\theta\cos\theta + 1 = 1$

$2\sin\theta\cos\theta = 0$

$\sin(2\theta) = 0$

General solutions:

$\Rightarrow 2\theta = 0 + 2k\pi$, k any integer and $2\theta = \pi + 2k\pi$, k any integer

$\theta = 0 + k\pi$, k any integer and $\theta = \frac{\pi}{2} + k\pi$, k any integer

On the interval $[0, 2\pi)$, the solutions appear to be $\theta = 0$, $\theta = \frac{\pi}{2}$, $\theta = \pi$, and $\theta = \frac{3\pi}{2}$.

Check your answers, since squaring both sides may introduce extraneous roots.

$\sin\theta + \cos\theta = 1$

For $\theta = 0$,

$\sin 0 + \cos 0 = \frac{y}{r} + \frac{x}{r} \qquad 0 : (1, 0)$

$= \frac{0}{1} + \frac{1}{1}$

$= 0 + 1$

$= 1$ ✓

$\sin\theta + \cos\theta = 1$

For $\theta = \frac{\pi}{2}$,

$\sin\frac{\pi}{2} + \cos\frac{\pi}{2} = \frac{y}{r} + \frac{x}{r} \qquad \frac{\pi}{2} : (0, 1)$

$= \frac{1}{1} + \frac{0}{1}$

$= 1 + 0$

$= 1$ ✓

$\sin\theta + \cos\theta = 1$

For $\theta = \pi$,

$\sin\pi + \cos\pi = \frac{y}{r} + \frac{x}{r} \qquad \pi : (-1, 0)$

$= \frac{0}{1} + \frac{-1}{1}$

$= 0 + -1$

$= -1$

You wanted 1, not -1.

So, discard $\theta = \pi$.

$\sin\theta + \cos\theta = 1$

For $\theta = \frac{3\pi}{2}$,

$\sin\frac{3\pi}{2} + \cos\frac{3\pi}{2} = \frac{y}{r} + \frac{x}{r} \qquad \frac{3\pi}{2} : (0, -1)$

$= \frac{-1}{1} + \frac{0}{1}$

$= -1 + 0$

$= -1$

You wanted 1, not -1.

So, discard $\theta = \frac{3\pi}{2}$.

Thus, on the interval $[0, 2\pi)$, the solutions are $\theta = 0$ and $\theta = \frac{\pi}{2}$.

All material has been taken from *Precalculus,* by M. Sullivan, 10th Edition